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On the Geometry of Semigroup Presentations

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1. INTRODUCTION

Given a semigroup presented by generators and defining relations, one can pose problems of a sort analogous to standard questions of combinatorial group theory. For instance, we can ask whether the semigroup has solvable *word problem*, or whether the semigroup can be embedded in some specified kind of algebraic object, e.g. a group. A result of the latter kind, due to Adjan [1], states that a semigroup is embeddable in a group provided that it has a finite presentation which is cycle-free (in a sense which we shall define later). In the present paper we develop *geometric* machinery for approaching such problems of "combinatorial semigroup theory." The geometric objects which we use are closely related to the "cancellation diagrams" of group theory, as introduced by Lyndon [3]. We apply the machinery to obtain a new proof of Adjan's embedding theorem, with the hypothesis of finiteness of the presentation removed. A subsequent paper will apply the same methods to obtain positive solutions of word problems for classes of semigroups considered by Grindlinger [2] and by Tartakovskii and Stender [8].

The use of two-dimensional geometric topology in combinatorial group theory has venerable beginnings, going back at least as far as Dehn's use in 1912 of embeddings of Cayley diagrams in the plane to solve the word and conjugacy problems for the fundamental groups of closed two-manifolds. A recent revival of the geometric approach stems from the 1966 paper of Lyndon cited above, and independent work of Weinbaum [9] in the same year. The method pioneered by Lyndon and developed by Schupp and others has become a powerful tool in small cancellation theory and other areas of combinatorial group theory; the reader may consult Lyndon and Schupp [4] for details and further references. The essence of the method, in its simplest form, is to construct, for each relation which is derivable from a set of defining relators for a given group, a *diagram* which "models" the derivation of the relation. The diagram is a labeled finite complex or graph embedded in the Euclidean plane. Combinatorial hypotheses on the set of defining relators yield information concerning the diagrams which can arise and thence information about the group. This approach has been used to solve word and conjugacy problems, and has been extended to the study of free products and free products with amalgamation.

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The diagrams which we employ here for semigroups are finite *directed* graphs (digraphs) embedded in the plane, with edges labeled by elements of a free semigroup S with basis X. Given a set R of defining relations over X, we shall define a class of diagrams, each corresponding to a pair of elements of S, such that a pair (u, v) corresponds to some diagram in the class iff (u, v) belongs to the congruence in S generated by R. A diagram for (u, v) models in a very direct way the usual, purely combinatorial process of transforming u to v by a finite succession of steps using the defining relations. The advantage of the diagram method, at least in the applications which we have investigated, seems to be that the geometric model makes relationships explicit and "visible" which tend to be hidden in the combinatorial approach. A fundamental difference between the present methods and those used for groups is the role played by the *directedness* of the underlying graphs, imposed by the asymmetry resulting from the absence of inverses.

We shall make free use throughout of the Jordan Curve Theorem and related elementary results of plane geometric topology. The reader is referred to Ore [6] for background.

In Section 2 we develop the rudiments of a theory of directed plane graphs, involving relationships between sources, sinks, and directed cycles. Our results here are elementary consequences of topological properties of the plane, although we are not aware of their prior existence in the literature. Group-diagrams and semigroup-diagrams are defined and compared in Section 3. In Section 4 we apply group- and semigroup-diagrams to obtain Adjan's embedding theorem and a related cancellation result.

2. Maps

A *directed map*, or simply a *map*, in the Euclidean plane Π is a finite, connected, directed graph M whose vertices are points of Π and whose edges are simple directed arcs, so that the endpoints of an edge are vertices, and an edge is not incident elsewhere with a vertex or another edge. Loops and parallel edges (edges with the same initial vertex and the same terminal vertex) are permitted. Each loop of M has an assigned orientation, which may be either positive ("counterclockwise") or negative. An edge indicent with a vertex P is an *in*-edge or *out*-edge at P according as P is its terminal or initial vertex. A *source* of M is a vertex without in-edges; a *sink* has no out-edges.

As usual, α^{-1} denotes the arc obtained by reversing the orientation of a directed arc α. For arcs α and β, $\alpha\beta$ denotes their concatenation in the standard sense, which is defined iff the terminal point of α coincides with the initial point of β. An arc

$$\pi = \alpha_1\alpha_2 \cdots \alpha_n \qquad (n \geqslant 1) \tag{1}$$

in which α_i is either e_i or e_i^{-1} for some edge of a map M, is a *walk* of M of *length* $|\pi| = n$. For convenience we regard each vertex P of M as a *trivial* walk of length zero running from P to P, with the convention that $P\pi = \pi Q = \pi$ whenever the walk π runs from P to Q. The walk (1) is *directed* if $\alpha_i = e_i$ for each i. A concatenation $\pi = \alpha\sigma\beta$ of walks determines the walk σ as a *segment* of π at *location* k, where $k = |\alpha| + 1$. The segment is *proper* if either α or β is non-trivial.

A walk is a *trail* if it intersects itself in at most vertices, a *path* if it is a simple open arc, and a *cycle* if it is a simple closed curve. Each of these notions can be characterized combinatorially in terms of the edges e_i in (1). For instance, π is a trail iff $e_i \neq e_j$ whenever $i \neq j$.

A map M separates the plane into finitely many connected components, all of which are open sets and exactly one of which, the *exterior* of M, is unbounded. Each bounded component is a *region* of M. The connectedness of M insures that every region is simply connected, and so is homeomorphic to the open disk. The *boundary* ∂M of M is the subgrouph of M whose vertices and edges are contained in the topological closure of the exterior of M, an edge or vertex being termed a *boundary* or *interior* edge or vertex according as it does or does not belong to ∂M. Likewise, the boundary ∂D of a region D consists of the edges and vertices contained in the closure of D. A closed walk of minimal length which includes all the edges on ∂D or ∂M is a *boundary walk* of ∂D or ∂M. A region is *cyclic* if some boundary walk of it is a *directed* cycle.

A connected subgraph H of a map M is itself a map, each of whose regions contains as a subset one or more regions of M. Each such region of M is said to be *enclosed by* H. A vertex of M is *enclosed by* H if it is a member of some region of H. An edge e of M is *enclosed by* H if every point of e, except possibly its endpoints, belongs to some region of H. If every region of H is a region of M, H is a *submap* of M. If H, considered as a map, has no interior edges or vertices, H is of *boundary type*. For example, ∂M is of boundary type. The map consisting of all vertices and edges of M which are either part of or enclosed by H is a submap of M, called the submap *bounded by* H. In this connection, there is no harm in identifying a cycle γ of M with the subgraph of edges and vertices belonging to γ, and speaking of the submap bounded by γ.

Let H be a boundary type subgraph of the map M. A walk π is *strictly interior* to H if every edge of π is enclosed by H and, in the sequence of successive vertices $P_0, P_1, \ldots, P_n$ belonging to π, none of $P_1, P_2, \ldots, P_{n-1}$ belong to H. A directed trail strictly interior to H is *maximal* with respect to H if it is not a proper segment of any other such directed trail. An *inner transversal* of H is a path strictly interior to H having both its endpoints on H. A directed inner transversal of H is clearly *a* maximal directed trail with respect to H.

Let M be a map, H a boundary type subgraph of M, and σ a directed trail strictly interior to H which is maximal with respect to H. Then the initial vertex P of σ either belongs to H, is a source of M enclosed by H, or belongs to a directed cycle which is a segment of σ. For if the first and third alternatives do not hold, then P is interior to H, and no segment of σ ends at P. Hence no edge e can end at P, since otherwise $e\sigma$ would be a directed trail interior to H, violating the maximality of σ. An obvious dual statement holds for the terminal vertex of σ.

LEMMA 2.1. *Every directed cycle of a map M encloses a source, a sink, or a cyclic region of M.*

Proof. Let γ be a directed cycle of M which encloses no sources or sinks. We shall show by induction on the number n of edges enclosed by γ that γ must enclose a cyclic region. If $n = 0$, then γ is itself the boundary of a cyclic region. Suppose $n > 0$; we assert that γ properly encloses another directed cycle. The desired conclusion will then follow by induction. For this, let σ be a directed trail which is strictly interior to γ and maximal with respect to γ. If one of the endpoints of σ does not lie on γ, then, since it cannot be a source or a sink, some segment of σ is a directed cycle properly enclosed by γ. Thus if no segment of σ is a directed cycle, σ is a directed inner transversal of γ. The endpoints of σ then divide γ into two directed paths, one of which, say α, runs from the terminal point to the initial point of σ. The directed cycle $\alpha\sigma$ is properly enclosed by γ. ∎

A *separating vertex* of a graph is a vertex whose removal disconnects the graph.

LEMMA 2.2. *Let M be a map having no cyclic regions and no interior sources or sinks. Then*:

(a) *M has no directed cycles.*

(b) *Every directed path strictly interior to a subgraph H of boundary type can be extended to a directed inner transversal of H.*

(c) *Every separating vertex of M is on ∂M.*

(d) *The boundary of every region of M is a cycle.*

Proof. Statement (a) is an immediate consequence of 2.1. Hence any maximal extension with respect to a subgraph H of a directed path strictly interior to H is a path which, since M has no interior sources or sinks, is an inner transversal of H, establishing (b). Since ∂M is connected, it suffices for (c) to show that given a pair P and Q of distinct interior vertices of M, there is a path running from Q to ∂M which does not contain P. By (b), there is a directed inner transversal $\alpha\beta$ of ∂M containing Q, where α runs from ∂M to Q and β from Q to ∂M; one of α and β does not contain P. For a proof that (c) implies (d), the reader is referred to Ore [6, Chapter 1]. ∎

A cycle γ of M and a vertex P on γ determine an *interior angle* θ of M *at* P whose *sides* are the edges of γ incident with P. An edge enclosed by γ is said to be *included in* θ if it is incident with P; a region is *included in* θ if it is enclosed by γ

and its boundary contains P. An interior angle at P is a *source-angle* (respectively, *sink-angle*) if its sides and included edges are all out-edges (resp. in-edges) at P.

A map M, or a region of a map M, is *two sided* if it has a boundary walk of the form $\omega\alpha^{-1}$, where α and ω are non-trivial directed trails. A cycle of M is *two sided* if it is of the form $\omega\alpha^{-1}$, where α and ω are non-trivial directed paths. We can choose notation so that $\omega\alpha^{-1}$ is positively oriented; then α and ω are uniquely determined, and are the *left* and *right sides* of the map, region, or cycle. A two sided map, region, or cycle has a unique *initial* vertex P (the common initial vertex of its left and right sides) and *terminal* vertex Q; its *initial* edges (which may coincide) are incident with P and its *terminal* edges incident with Q.

The map M is *regular* if (1) every region of M is two sided, (2) M is two sided, and (3) M has no interior sources or sinks. Note that a regular map satisfies 2.2. In addition, we have:

LEMMA 2.3. *Let M be a regular map. Then:*

(a) *Every two sided submap of M is regular.*

(b) *Every pair of distinct edges with common initial (resp. terminal) vertex P are the initial edges (resp. terminal edges) of a two-sided cycle γ of M. The angle at P determined by γ is a source-angle (resp. sink-angle) at P.*

(c) *The initial vertex O and the terminal vertex I of M are, respectively, the unique source and the unique sink of M.*

(d) *Every vertex and edge of M lies on a directed path running from O to I.*

Proof. Statement (a) holds since clauses (1) and (3) of the definition of regularity are inherited by submaps. Statement (d) is obvious for vertices on the boundary of M, since M is two-sided. If P is an interior vertex, then by 2.2(b), P lies on some directed inner transversal of M, which can then be extended to a directed path from O to I. This establishes (d). Hence given a vertex P, any pair of distinct edges starting at P form the initial edges of some two sided cycle γ of M. Supposing that γ does not determine a source-angle at P, there exists by 2.2(b) a directed inner transversal π of γ, running from a vertex Q on a side β of γ to the initial vertex P of γ. Letting β' be the initial segment of β running from P to Q, $\beta'\pi$ is a directed cycle, contradicting 2.2(a). This proves (b). For (c), it is clear from (d) that if O is a source, it is the unique source of M. If the initial edges of the sides of M coincide, O is clearly a source. If they do not, we use (b) to infer that O is a source. By an obvious dual argument, I is the unique sink of M. ∎

It will be useful to describe more explicitly the structure of two sided maps. Let α and ω be the left and right sides of the two-sided map M; assume that α and ω are paths. Then they are unique concatenations $\alpha = \alpha_1\alpha_2 \cdots \alpha_n$ and $\omega = \omega_1\omega_2 \cdots \omega_n$ of non-trivial segments, where for each i either $\alpha_i = \omega_i$ or

$\omega_i\alpha_i^{-1}$ is a positively oriented cycle, and for each $i \geqslant 2$ either $\omega_{i-1}\alpha_{i-1}^{-1}$ or $\omega_i\alpha_i^{-1}$ is a cycle. Each $\omega_i\alpha_i^{-1}$ is exterior to all the rest. The submap bounded by $\omega_i\alpha_i^{-1}$ is called the ith *simple component* of M. Thus each simple component of M is either a directed path or is bounded by a two sided cycle.

3. Diagrams

Let $F = F(X)$ be the free group on (finite or infinite) basis X. For each basis element x, the elements x and x^{-1} will be called *letters*; then each element of F is the value of some word

$$y_1 y_2 \cdots y_n \qquad (n > 0;\ \text{each } y_i \text{ a letter}) \tag{1}$$

over X, and is the value of a unique reduced word. A word $y_{p+1}y_{p+2} \cdots y_q$ $(1 \leqslant p \leqslant q \leqslant n)$ is a *segment* of the word (1) at *location* p. The sub-semigroup $S(X)$ of F generated by X is the free semigroup with basis X; each element of $S(X)$ is the value of a unique *positive* word $x_1x_2 \cdots x_n$ $(n > 0;\ x_i \in X)$ over X. We say that words u, v over X are *literally* equal, denoted $u \equiv v$, if they are identical regarded as sequences of letters, and *freely* equal if they define the same element of F. The length of a word u is denoted $|u|$, the empty word by Λ. A word is *trivial* if it is freely equal to Λ. The group presented by generators X and relators U, where U is a set of words over X, is denoted $\langle X;\ U\rangle$. It is the quotient of F by the normal subgroup generated by U; thus every word over X defines a unique element of $\langle X;\ U\rangle$, and every element of $\langle X;\ U\rangle$ is defined by some word over X.

A *diagram* M over X is a map equipped with a labeling function φ which assigns to each edge e of M a positive word $\varphi(e)$ over X. The labeling is extended to all walks of M by the rules $\varphi(P) \equiv \Lambda$, $\varphi(e^{-1}) \equiv \varphi(e)^{-1}$, and $\varphi(\alpha\beta) \equiv \varphi(\alpha)\,\varphi(\beta)$. The label on each non-trivial directed walk is thus a positive word. For each region D of M, the label of any boundary walk of D is a *label of* D. A *label of* M is the label of any boundary walk of M.

Let U be a set of words over X, and M a diagram over X with label w such that each region of M has a label which is conjugate in F to some element of U or to the empty word. Then it is known that w defines the identity element of $\langle X;\ U\rangle$ (see Schupp [7, Theorem 3.1]). A converse of this holds—if w belongs to the normal subgroup of F generated by U, then there is a diagram having a boundary label which defines a conjugate in F of w and whose regions satisfy the above labeling requirement. Moreover, one can require that the labels on the boundary and regions of the diagram be cyclically reduced (see Lyndon [3, Lemma 3.1]).

We will not prove here the facts cited in the preceding paragraph. However, one technique employed in proving them can be adapted to obtain a closely related result which is of use in the sequel. Let M be a diagram over X. Lyndon

[3, p. 216] has shown that if M has a label containing a proper segment $y^{-1}y$, y a letter, then M can be modified to obtain a new map M' having boundary label that of M with the segment $y^{-1}y$ deleted, such that every region D' of M' corresponds to a region of M with the same label as D', different regions of M' corresponding to different regions of M. The modification to produce M' consists of continuously deforming M so as to bring oppositely labeled consecutive segments of ∂M into coincidence, or in some cases deleting submaps of M. The same construction can be applied to any region of M to eliminate proper segments $y^{-1}y$ of its label. Supposing that some region D of M has a non-reduced boundary label we may, by iterating the construction, modify M to produce a new diagram M' with a region D' corresponding to D and having boundary label either a cyclically reduced word or a trivial relator $x^{-1}x$ for some $x \in X$. In the latter case D' is two-sided, the sides α and ω each bearing label x. Then we can modify M' so as to eliminate D' altogether, by deforming the portion of M' lying in some small neighborhood of ω to bring ω into coincidence with α. A sufficient number of applications of this construction yields:

LEMMA 3.1. *Let U be a set of words over the basis X of the free group F. Let M be a diagram over X with label w, and $D_1, D_2, \ldots, D_n$ a sequence of regions of X such that each D_i has label freely equal to a conjugate in F of some $u_i \in U$, the remaining regions of M all having label freely equal to Λ. Then there is a diagram M' over X having no more than n regions, such that the boundary of M' is a cyclically reduced conjugate of w, and the label on each region is a cyclically reduced conjugate of some u_i.*

We now fix a (finite or infinite) set R of unordered pairs of positive words over X; each pair $\{r, s\} \in R$ is called a *defining relation* with respect to R. Then $[X; R]$ denotes the semigroup defined by the generators X and relations R. That is, $[X; R]$ is the quotient of the free semigroup $S(X)$ by the congruence generated by R, two positive words u and v over X having the same image in $[X; R]$ iff there is a sequence

$$u_0 \to u_1 \to \cdots \to u_n \qquad (n \geqslant 0) \tag{2}$$

with $u_0 \equiv u$ and $u_n \equiv v$. The notation $w \to w'$ means that the word w' is *directly derived* from w by replacement of a segment r of w by some s with $\{r, s\} \in R$. The sequence (2) will be called a *regular derivation* of *length* n for the pair (u, v). We shall write $u = v(R+)$ to indicate that a regular derivation exists for (u, v).

The group $\langle X; R\rangle$ presented by generators X and relations R is defined to be the group $\langle X; U\rangle$, where $U = \{sr^{-1} \mid \{r, s\} \in R\}$. We write $u = v(R)$ if words u and v have the same image in $\langle X; R\rangle$. For positive words u and v, the relation $u = v(R+)$ implies $u = v(R)$. The *natural* homomorphism of $[X; R]$ into $\langle X; R\rangle$ maps the image in $[X; R]$ of each positive word to its value in $\langle X; R\rangle$.

This homomorphism is an embedding iff $u = v(R)$ implies $u = v(R+)$ for all positive words u, v.

The pair (X, R) is said to be a *positive presentation* of the semigroup $[X; R]$ and the group $\langle X; R\rangle$.

We now define two kinds of diagrams which will be fundamental objects of study in the next section.

A *group-diagram*, or *g-diagram*, *over* (X, R) for a word w over X is a diagram M over X with the properties:

(G1) Each region of M is labeled by a cyclically reduced conjugate (in the free group F) of the word sr^{-1} for some defining relation $\{r, s\} \in R$.

(G2) The boundary of M is labeled by a cyclically reduced conjugate of w.

A *semigroup-diagram*, or *s-diagram*, over (X, R) for a pair (u, v) of positive words is a diagram M over X with the properties:

(S1) Each region of M is labeled by the word sr^{-1} for some defining relation $\{r, s\} \in R$.

(S2) The boundary of M is labeled by the word vu^{-1}.

(S3) There are no interior sources or sinks of M.

Note that the conditions (S1)–(S3) imply that M is regular in the sense of the preceding section. The conditions (G1) and (G2) imply that every region of a g-diagram is two sided or cyclic. In view of our previous remarks, a g-diagram over (X, R) exists for a word w iff w defines the identity element of the group $\langle X; R\rangle$. An analogous connection holds between s-diagrams and the semigroup $[X; R]$, as we now demonstrate.

THEOREM 3.2. *Let u and v be positive words over X. There is a regular derivation of length n of the relation $u = v(R+)$ iff there is an s-diagram over (X, R) for the pair (u, v) having exactly n regions.*

Proof. Assume that M is an n-region s-diagram for the pair (u, v), with left side α labeled $u \equiv r_1 r_2 \cdots r_m$ and right side ω labeled $v \equiv s_1 s_2 \cdots s_m$, where $s_i r_i^{-1}$ is the label on the boundary of the ith simple component M_i of M traversed in the positive sense starting at the initial vertex of M_i. If M has no interior edges, then for each i either $r_i \equiv s_i$ or ∂M_i is a cycle and $\{r_i, s_i\}$ is a defining relation. Then $u = v(R+)$ via a regular derivation consisting of n non-overlapping replacements of the form $pr_iq \to ps_iq$. If M has an interior edge e, we extend e by 2.3(d) to a directed path π running from the initial vertex to the terminal vertex of M. The submaps M' bounded by $\pi\alpha^{-1}$ and M'' bounded by $\omega\pi^{-1}$ are regular by 2.3(a), and each has fewer interior edges than does M. By induction we may assume that regular derivations exist of the relations $u = w(R+)$ and $w = v(R+)$ of lengths equal to the number of regions of M'

and M'' respectively, where w is the label on π. By concatenating these derivations we obtain a regular derivation of $u = v(R+)$ of length n.

For the converse, we suppose that a regular derivation $u_0 \to u_1 \to \cdots \to u_n$ is given for (u, v). If $n = 0$, then $u \equiv v$, and M can be taken as a simple directed open arc with label u. If $n > 0$, we may assume by induction that N is an s-diagram for the pair (u, u_{n-1}) with $n - 1$ regions. Now $u_{n-1} \equiv prq$ and $u \equiv psq$ for some defining relation $\{r, s\}$. By introducing at most two extra vertices on the right side ω of N, we have $\omega = \beta\rho\gamma$, where β, ρ, γ have labels p, r, q respectively. Let σ be a simple arc with label s running from the initial to the terminal vertex of ρ in the exterior of N, chosen so that the cycle $\sigma\rho^{-1}$ is positively oriented. Then the diagram M obtained from N by adjoining σ as an edge is the desired s-diagram for (u, v).

The method of proof of the "if" part of 3.2 is easily adapted to prove a related result which may be of interest in its own right. Let M be a regular map whose edges are labeled by elements of some semigroup S, the labeling being extended to directed paths of M in the obvious way. Then if for each region of M the labels on the sides of that region are equal in S, it follows that the labels on the sides of M are also equal.

We close this section with two examples.

EXAMPLE 1. In the semigroup defined by generating set $X = \{a, b, c, d, e\}$ and relation set (written in equation form) $R = \{ad = bd, c = de\}$, the relation $ac = bc(R+)$ has (shortest possible) regular derivation

$$ac \to ade \to bde \to bc$$

of length 3. The corresponding s-diagram is displayed in Figure 1.

EXAMPLE 2. Malcev [5] gave the following as an example of a cancellation semigroup which is not embeddable in any group. Let $X = \{a, b, c, d, x, y, u, v\}$, $R = \{ax = by, cx = dy, au = bv\}$. It is clear by inspection that $cu \neq dv(R+)$;

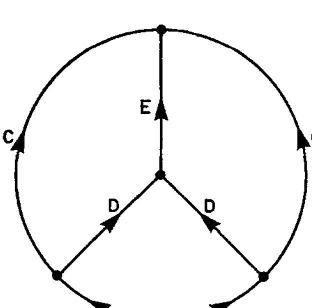

FIG. 1. An s-diagram for Example 1.

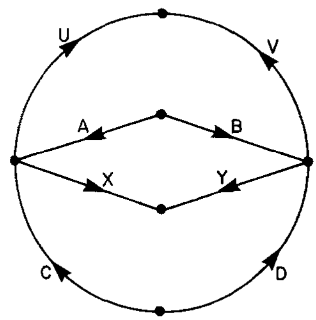

FIG. 2. A g-diagram for Example 2.

hence no s-diagram for this relation exists. However, we have $cu = dv(R)$, as can be seen either by a direct computation or by examining the g-diagram of Figure 2. Note that although the map and each region of it is two-sided, the map is not regular since it contains an interior source and an interior sink.

4. CANCELLATION AND EMBEDDING

Throughout this section, (X, R) denotes a fixed positive presentation. All diagrams mentioned are understood to be diagrams over X, all g- and s-diagrams being with respect to (X, R).

We wish to investigate certain classes of groups and semigroups whose definitions are conveniently expressed in graph-theoretic terms. For this purpose, we introduce two undirected graphs determined by (X, R), the *left graph* $LG(X, R)$ and the *right graph* $RG(X, R)$. Both graphs have vertex set X and edge set R. In $LG(X, R)$ a defining relation $\{r, s\}$ joins the initial letters of r and s; in $RG(X, R)$ it joins their terminal letters. Loops and parallel edges may occur in either graph. Left and right graphs of positive presentations were considered by Adjan [1].

A *walk* of *length* $m \geqslant 0$ in an undirected graph is determined by a sequence of vertices $a_0, a_1, \ldots, a_m$ and edges $f_1, \ldots, f_m$ such that $\{a_{i-1}, a_i\}$ is the set of edges indicent with f_i, $1 \leqslant i \leqslant m$. This requirement uniquely determines $a_1, a_2, \ldots, a_m$, given a_0 and the f_i. The walk is *closed* if $a_0 = a_m$ and is a *cycle* if it is closed, $m \geqslant 1$, and the vertices $a_1, \ldots, a_m$ are all different. Walks and cycles in $LG(X, R)$ are called *left walks* and *left cycles* of (X, R), *right walks* and *right cycles* being defined analogously with respect to $RG(X, R)$. If (X, R) has no left or right cycles it is *cycle-free*, and (X, R) is *left* or *right reduced* according as (X, R) has no left or right cycles of length one.

The main goal of this section is to use diagrams to obtain a non-computational proof of the theorem, due to Adjan [1], that a semigroup having a cycle-free positive presentation is embeddable in a group. We also derive some sufficient conditions for cancellation properties to hold in a semigroup.

A pair of regions D, D' of a diagram M is *inversely labeled* if their boundaries share an edge e whose deletion from M consolidates D and D' into a single region with boundary label which is freely equal to the empty word. The edge e is an *interface* of D and D'.

LEMMA 4.1. *Suppose that (X, R) has no left (resp. right) cycles and that θ is an interior source-angle (resp. sink-angle) of a g-diagram M such that the labels on the sides of θ have the same initial (resp. terminal) letter. Then there is a pair of inversely labeled regions included in θ.*

Proof. We may assume that (X, R) has no left cycles and that θ is a source-angle. Let $D_1, D_2, \ldots, D_k$ be the regions included in θ listed in negative cyclic order, that is, the order in which they are encountered when traversing in the negative sense an arc of a small circle with center at P whose ends are on the sides of θ. No D_i is a cyclic region, for otherwise some boundary edge of D_i would end at P, and θ would not be a source-angle. By the choice of ordering of the D_i, the right side of D_{i-1} and the left side of D_i share a common initial edge; let x_i be the initial letter of the label of this edge, and define $x_0 \equiv x_k$ to be the common initial letter of the labels on the sides of θ. The labels on the left and right sides of D_i are positive words p_i and q_i with distinct terminal letters (since the word $q_i p_i^{-1}$ is cyclically reduced), and for some positive or empty word t_i, $\{p_i t_i, q_i t_i\}$ is a defining relation. We denote this defining relation by f_i. The sequences $x_0, x_1, \ldots, x_k$ and $f_1, \ldots, f_k$ constitute a closed left walk of (X, R). Choose indices i and j, $1 \leqslant i < j \leqslant k$, such that $x_{i-1} \equiv x_j$, with $j - i$ as small as possible. Since (X, R) has no left cycles, $j = i + 1$ and the defining relations f_i and f_j are identical. Hence

$$p_i t_i \equiv q_j t_j \qquad \text{and} \qquad q_i t_i \equiv p_j t_j \tag{1}$$

and we may assume, without loss of generality, that t_i is a terminal segment of t_j, so that $t_j \equiv z t_i$ for some positive or empty word z. The relations (1) then become

$$p_i \equiv q_j z \qquad \text{and} \qquad q_i \equiv p_j z \tag{2}$$

after cancellation of t_i. Let u be the label of the common initial edge e of D_i and D_j, so that $q_i \equiv uv$ and $p_j \equiv uw$. From the second relation in (2), it follows that $v \equiv wz$. Deletion of e from M consolidates D_i and D_j into a single region with boundary label $q_j w^{-1} v p_i^{-1} \equiv q_j w^{-1} w z z^{-1} q_j^{-1}$, which is freely equal to the empty word. Hence D_i and D_j are the required inversely labeled regions. ∎

COROLLARY 4.2. *If (X, R) has no left (resp. right) cycles, and P is an interior* *source (resp. sink) of the g-diagram M, then there is a pair of inversely labeled regions incident with P.*

Proof. We may list the set of regions of M having P as a boundary point in negative cyclic order and apply the same argument as in 4.1. ∎

Given a presentation (X, R) and a g-diagram M for w which contains a pair of inversely labeled regions, we may remove their interface and apply 3.1 to obtain a g-diagram for w having at least two fewer regions than M. Iterating this, we obtain:

THEOREM 4.3. *If (X, R) has no left (resp. right) cycles, and M is a g-diagram for a word w, then there exists a g-diagram M' for w which contains no more regions than does M, has no interior sources (resp. sinks), and in which the labels on the sides and included edges of each interior source-angle have intitial (resp. terminal) letters which are all different. If (X, R) is cycle-free, then M' can be chosen to satisfy both sets of conditions simultaneously.*

A g-diagram satisfying the first set, second set, or both sets of conditions of 4.3 will be called *source-reduced, sink-reduced,* or *reduced,* according to the case. A close analogue of 4.3 is valid is valid for s-diagrams, but a separate proof is needed—the deformations used to eliminate regions with trivial label can in general introduce cyclic regions or interior sources or sinks. We base the argument instead on the fact that an s-diagram cannot have directed cycles.

THEOREM 4.4. *If (X, R) has no left (resp. right) cycles, and M is an s-diagram for (u, v), then there exists an s-diagram M' for (u, v) having no more regions than does M, such that the labels on every pair of distinct co-initial (resp. co-terminal) edges of M' have distinct initial (resp. terminal) letters. If (X, R) is cycle-free, then M' can be taken to satisfy both conditions simultaneously.*

Proof. Suppose that (X, R) has no left cycles and that there is at least one vertex P of M which possesses two out-edges bearing labels with the same initial letter. Since by 2.2(a) M has no directed cycles, we may choose P *extremal* with this property, so that no other vertex on any directed path starting at P possesses two such out-edges. From 2.3(b) we infer that e and f are the sides of a source-angle θ at P. As in the proof of 4.1, there must exist regions D, D' included in θ which are labeled by the same defining relation, and so that the right side β of D and the left side γ of D' are identically labeled and share a common initial edge. Now β and γ must coincide, for otherwise they would possess distinct, proper, identically labeled terminal segments with the same initial vertex, violating the choice of P as extremal. Thus deletion of β from M consolidates D and D' into a single two-sided region E. We now deform the portion of M lying in a small neighborhood of the right side of E to bring it into coincidence with the left side of E, reducing the number of regions of M by two.

Iteration of this process must eventually terminate with an s-siagram with the desired properties. ▮

As with g-diagrams, we apply the terms *source-reduced*, *sink-reduced*, and *reduced* to s-diagrams satisfying the corresponding conditions stated in 4.4.

COROLLARY 4.5. *If (X, R) has no left (resp. right) cycles and there is a regular derivation of length n of the relation $uv = uw(R+)$ (resp. $vu = wu(R+)$), then there is a regular derivation of $v = w(R+)$ of length not exceeding n. In particular, $[X; R]$ has the left (resp. right) cancellation property.*

Proof. Assuming that (X, R) has no left cycles and that $uv = uw(R+)$ via a regular derivation of length n, there exists by 4.4 a source-reduced s-diagram M for the pair (uv, uw) having no more than n regions. The left and right sides of M have initial segment ρ and π each labeled u; since M is source-reduced, ρ and π must coincide. Deletion of ρ from M yields an s-diagram for (v, w) having the same number of regions as M. ▮

The next result contains Adjan's theorem.

THEOREM 4.6. *Suppose that (X, R) is cycle-free, u and v are positive words, and that M is a g-diagram for the word vu^{-1} containing n regions. Then $u = v(R+)$ via a regular derivation of length not exceeding n. In particular, the natural homomorphism of $[X; R]$ into $\langle X; R\rangle$ is an embedding.*

Proof. We may take M to be a reduced g-diagram. Since (X, R) is left and right reduced, the word sr^{-1} is cyclically reduced as it stands, for every defining relation $\{r, s\}$. Hence each region of M is two sided. Since moreover M contains no interior sources or sinks, M contains no directed cycles, by 2.1. In particular, the boundary of M, being labeled by a cyclically reduced conjugate of vu^{-1}, is two sided. Hence M is an s-diagram, with sides labeled by words u', v', where $u \equiv pu'q$ and $v \equiv pv'q$ for some positive or empty words p and q. We construct the desired s-diagram for (u, v) by attaching an arc with label p to the initial point of M and an arc with label q to the terminal point of M. ▮

We remark that Adjan proved the embedding theorem under the assumption that X and R are both finite sets. The present method does not entail this restriction.

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